

Radio Labelings of Distance Graphs

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Abstract

A radio k -labeling of a connected graph G is an assignment c of non negative integers to the vertices of G such that

$$|c(x) - c(y)| \geq k + 1 - d(x, y),$$

for any two vertices x and y , $x \neq y$, where $d(x, y)$ is the distance between x and y in G . In this paper, we study radio labelings of distance graphs, i.e., graphs with the set \mathbb{Z} of integers as vertex set and in which two distinct vertices $i, j \in \mathbb{Z}$ are adjacent if and only if $|i - j| \in D$.

Keywords: graph labeling, radio k -labeling number, distance graph.

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1 Introduction

Let G be a connected graph and let k be an integer, $k \geq 1$. The distance between two vertices u and v of G is denoted by $d_G(u, v)$ (or simply $d(u, v)$) and the diameter of G is denoted by $\text{diam}(G)$. A *radio k -labeling* c of G is an assignment of non negative integers to the vertices of G such that

$$|c(u) - c(v)| \geq k + 1 - d(u, v),$$

for every two distinct vertices u and v of G . The span of the function c , denoted by $\text{rl}_k(c)$, is $\max\{c(x) - c(y) : x, y \in V(G)\}$. The *radio k -labeling number* $\text{rl}_k(G)$ of G is the minimum span among all radio k -labelings of G .

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The study of radio k -labelings was initiated by Chartrand et al. [1] and is motivated by radio channel assignment problems with interference constraints. Quite few results are known concerning radio k -labelings. The radio k -labeling number for paths was first studied by Chartrand et al. [1], where lower and upper bounds were given. These bounds have been improved by Kchikech et al. [8].

Except the above, radio k -labelings have been investigated mainly for fixed values of k . A radio 1-labeling is a proper vertex-colouring and $\text{rl}_1(G) = \chi(G) - 1$. For $k = 2$, the radio 2-labeling problem corresponds to the well studied $L(2, 1)$ -labeling problem (see for instance [6, 9] and references therein). Large values of k (close to the diameter of the graph) were also considered for radio k -labelings. The interested reader is referred to surveys [5, 17] and recent papers [15, 18] for complementary results.

Notice that the authors of [1, 2, 3, 4] assume that the labels (colours) are positive. However, when speaking about labelings in relation with frequency assignment, it is more common to use non negative integers as labels. Thus the notation of the present paper follows the terminology of [7, 8, 11, 13, 14, 15, 18] in which vertices are labeled by non negative integers.

The (infinite) *distance graph* $G(\mathbb{Z}, D)$ with distance set $D = \{d_1, d_2, \dots, d_t\}$, where d_i being positive integers, has the set \mathbb{Z} of integers as vertex set, with two distinct vertices $i, j \in \mathbb{Z}$ being adjacent if and only if $|i - j| \in D$. The *finite distance graph* $G_n(D)$ is the subgraph of $G(\mathbb{Z}, D)$ induced by vertices $0, 1, \dots, n-1$. To simplify, if $D = \{d_1, d_2, \dots, d_t\}$, then $G(\mathbb{Z}, D)$ will also be denoted by $D(d_1, d_2, \dots, d_t)$ and $G_n(D)$ by $D_n(d_1, d_2, \dots, d_t)$. In [10, 19, 20], radio 2-labeling numbers have been determined only for some of the distance graphs (mainly 4-regular). The aim of this paper is to obtain bounds on the radio k -labeling number of some distance graphs in terms of k (and not depending on the order of the graph). We show that

$$\frac{t}{2}k^2 + \frac{1}{2} \leq \text{rl}_k(D(1, 2, \dots, t)) \leq \begin{cases} \frac{t}{2}k^2 + \frac{t}{2}k, & \text{when } k \text{ is odd,} \\ \frac{t}{2}k^2 + k, & \text{when } k \text{ is even.} \end{cases}$$

Then we prove the following bounds of the distance graphs $D(1, t)$ and $D(t-1, t)$ as subgraphs of $D(1, 2, \dots, t)$, $t \geq 2$.

$$\begin{aligned} \frac{t}{2}k^2 - P_2(t)k + P_3(t) &\leq \text{rl}_k(D(1, t)) \leq \frac{t}{2}k^2, & \text{for } t \geq 3 \text{ and odd } k, \\ \frac{t}{2}k^2 - P'_2(t)k + P'_3(t) &\leq \text{rl}_k(D(t-1, t)) \leq \frac{t}{2}k^2 + k - \frac{t+2}{2}, & \text{for } t \geq 3 \text{ and odd } k, \end{aligned}$$

where $P_i(t)$, $P'_i(t)$ denote polynomials of variable t of degree i .

Recall the known bounds for the radio k -labeling number of the most basic distance graph, i.e. the infinite path $P_\infty = D(1)$. It was shown [1, 8] that for any $n > 3$ and any $1 \leq k \leq n-3$,

$$\frac{k^2 + 4}{2} \leq \text{rl}_k(P_n) \leq \frac{k^2 + 2k}{2}, \text{ if } k \text{ is even,}$$

$$\frac{k^2 + 1}{2} \leq \text{rl}_k(P_n) \leq \frac{k^2 + 2k - 1}{2} \text{ if } k \text{ is odd,}$$

and conjectured [8] that the upper bound is the exact value of the radio k -labeling number when the length of the path is large enough.

2 Lower bounds

A classical method for finding a lower bound on the radio k -labeling number of a graph is to use the following relation with another graph parameter called the *upper traceable number* [16], denoted t^+ : for a graph G of order n and for a linear ordering $s : (x_1, x_2, \dots, x_n)$ of its vertices, let $d(s) = \sum_{i=1}^{n-1} d(x_i, x_{i+1})$. Then $t^+(G) = \max d(s)$, where the maximum is taken over all linear orderings s of vertices of G .

Theorem 1 ([7]). *For any integer $k \geq 1$, and any graph G of order n ,*

$$\text{rl}_k(G) \geq (n-1)(k+1) - t^+(G).$$

In order to find bounds for the upper traceable number of some distance graphs, the upper traceable number of the path (determined in [13], without using the above terminology) will be useful:

Lemma 1 ([13]). *For any integer $n \geq 2$,*

$$t^+(P_n) = \begin{cases} \frac{1}{2}n^2 - 1 & \text{if } n \text{ is even,} \\ \frac{1}{2}(n^2 - 1) - 1 & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 2. *Let G be a graph of order n with $V(G) = \{0, 1, \dots, n-1\}$. If there are positive real numbers α and β such that $d_G(i, j) \leq \frac{j-i+\alpha}{\beta}$ for any i and j satisfying $0 \leq i < j \leq n-1$, then*

$$t^+(G) \leq \frac{\frac{n^2}{2} + \alpha(n-1) - 1}{\beta}.$$

Proof. For a path P on the vertices $0, 1, \dots, n-1$, we have $d_P(i, j) = |j - i|$ for every $i, j \in \{0, 1, \dots, n-1\}$. Hence, for any ordering (x_1, x_2, \dots, x_n) of the vertices of G , we have

$$\sum_{i=1}^{n-1} d_G(x_i, x_{i+1}) \leq \sum_{i=1}^{n-1} \frac{|x_{i+1} - x_i| + \alpha}{\beta} = \sum_{i=1}^{n-1} \frac{d_P(x_i, x_{i+1}) + \alpha}{\beta} \leq \frac{t^+(P_n) + \alpha(n-1)}{\beta}.$$

Therefore, with Lemma 1, we obtain the desired inequality. \square

Proposition 1. For any positive integers $k \geq 1$ and $t \geq 2$,

$$rl_k(D(1, 2, \dots, t)) \geq \frac{t}{2}k^2 + \frac{1}{2}.$$

Proof. Let $n > t$ and let $G = D_n(1, 2, \dots, t)$. It is easily seen that for $j \geq i$,

$$d_G(i, j) = \left\lceil \frac{j-i}{t} \right\rceil \leq \frac{j-i+t-1}{t}.$$

Hence G satisfies the conditions of Lemma 2 with $\alpha = t-1$ and $\beta = t$ and we have

$$t^+(G) \leq \frac{\frac{n^2}{2} + (t-1)(n-1) - 1}{t} = \frac{n}{t}(\frac{n}{2} + t - 1) - 1.$$

Consequently, by Theorem 1, we obtain that

$$rl_k(G) \geq (n-1)(k+1) - \frac{n}{t}(\frac{n}{2} + t - 1) + 1 = n(k - \frac{n-2}{2t}) - k.$$

The right hand side of the inequality is maximized when $n = tk + 1$ and it gives

$$rl_k(D_{tk+1}(1, 2, \dots, t)) \geq (tk+1)(k - \frac{tk-1}{2t}) - k = \frac{t}{2}k^2 + \frac{1}{2t}.$$

As $D_{tk+1}(1, 2, \dots, t)$ is a subgraph of $D(1, 2, \dots, t)$ and since the radio k -labeling number is a natural number, we have $rl_k(D(1, 2, \dots, t)) \geq \lceil \frac{t}{2}k^2 + \frac{1}{2t} \rceil \geq \frac{t}{2}k^2 + \frac{1}{2}$ and the desired inequality is proved. \square

For the graphs $D(1, t)$, we can show a lower bound of the same order by using a similar argument. We will use the following statement.

Lemma 3 ([21]). The distance between two vertices i and j of $D(1, t)$ is $d(i, j) = \min(q + r; q + 1 + t - r)$, where $|i - j| = qt + r$, with $0 \leq r < t$.

Proposition 2. For any positive integers $t \geq 3$ and $k \geq \frac{t}{2}$,

$$rl_k(D(1, t)) \geq \frac{t}{2}k^2 - P(t)k + Q(t),$$

with $P(t) = \frac{t^2}{2} - t + \frac{1}{2}$ and $Q(t) = \frac{t^3}{8} - \frac{t^2}{2} + \frac{3t}{4} - \frac{1}{2}$.

Proof. Let n be a positive integer and let $G = D_n(1, t)$. Then, by Lemma 3, $d_G(i, j) = q + \min(r, t + 1 - r)$, for $j \geq i$ with $j - i = qt + r$ and $0 \leq r < t$. Thus

$$d_G(i, j) \leq \frac{j-i + \lceil \frac{t}{2} \rceil (t-1)}{t} \leq \frac{j-i + \frac{t+1}{2}(t-1)}{t}.$$

Hence G satisfies the conditions of Lemma 2 with $\alpha = \frac{t^2-1}{2}$ and $\beta = t$ and we have

$$t^+(G) \leq \frac{\frac{n^2}{2} - 1 + \frac{t^2-1}{2}(n-1)}{t} = \frac{(n-1)(n+t^2) - 1}{2t}.$$

Consequently, by Theorem 1, we obtain that

$$rl_k(G) \geq (n-1)(k+1) - \frac{(n-1)(n+t^2)-1}{2t} = (n-1)(k+1 - \frac{n+t^2}{2t}) + \frac{1}{2t}.$$

The right hand side of the inequality is maximized when $n = tk - \lceil \frac{t^2}{2} \rceil + t$ and since, by the hypothesis, $k \geq \frac{t}{2}$, we get $n \geq 1$. Reporting this value in the above inequation gives

$$rl_k(D_{tk - \lceil \frac{t^2}{2} \rceil + t}(1, t)) \geq (tk - \lceil \frac{t^2}{2} \rceil + t - 1)(k + 1 - \frac{tk - \lceil \frac{t^2}{2} \rceil + t + t^2}{2t}) + \frac{1}{2t}.$$

After simplification, in both cases t is odd and t is even, we obtain

$$rl_k(D_{tk - \lceil \frac{t^2}{2} \rceil + t}(1, t)) \geq \frac{t}{2}k^2 - (\frac{t^2}{2} - t + \frac{1}{2})k + \frac{t^3}{8} - \frac{t^2}{2} + \frac{3t}{4} - \frac{1}{2} + \frac{1}{2t},$$

which concludes the proof. \square

Corollary 1. *For any $k \geq 2$,*

$$rl_k(D(1, 3)) \geq \frac{3}{2}k^2 - 2k + \frac{5}{8}.$$

For the graphs $D(t-1, t)$, we first compute an upper bound on the distance between two vertices:

Lemma 4. *Let $t \geq 2$ be an integer, i, j a pair of vertices of the graph $G = D(t-1, t)$ and let $|i - j| = qt + r$, where $q, r \in \mathbb{N}$, $0 \leq r < t$. Then $d_G(i, j) \leq q + t$.*

Proof. Let i, j be two integers with $j \geq i$. If $0 \leq j - i \leq t/2$ then $d_G(i, j) \leq 2(j - i) \leq t$ since $j - i = (j - i)t - (j - i)(t - 1)$. If $t/2 < j - i \leq t - 1$ then $d_G(i, j) \leq 2(t - j + i) + 1 \leq 2t - t - 1 + 1 = t$ since $j - i = (t - j + i)(t - 1) - (t - 1 - j + i)t$. Now, if $j - i \geq t$ then $j - i = qt + r$, with $q, r \in \mathbb{N}$ and $0 \leq r < t$. Hence, with the above, $d_G(i, j) \leq q + t$. \square

Proposition 3. *For any positive integers $t \geq 3$ and $k \geq t$,*

$$rl_k(D(t-1, t)) \geq \frac{t}{2}k^2 - P(t)k + Q(t),$$

with $P(t) = t^2 - t + 1$ and $Q(t) = \frac{t^3}{2} - t^2 + \frac{3}{2}t - 1$.

Proof. Let n, t be integers, $t \geq 3$, and let $G = D_n(t-1, t)$. Then, for any integers i, j , $0 \leq i \leq j \leq n-1$, we have $d_G(i, j) \leq \lfloor \frac{j-i}{t} \rfloor + t \leq \frac{j-i+t^2}{t}$ by Lemma 4.

Hence G satisfies the conditions of Lemma 2 with $\alpha = t^2$ and $\beta = t$ and we have

$$t^+(G) \leq \frac{\frac{n^2}{2} - 1 + t^2(n-1)}{t} = (n-1)(t + \frac{n}{2t}) + \frac{n}{2t} - \frac{1}{t}.$$

Consequently, by Theorem 1, we obtain that

$$rl_k(G) \geq (n-1)(k+1) - (n-1)(t + \frac{n}{2t}) - \frac{n}{2t} + \frac{1}{t} = (n-1)(k+1 - t - \frac{n}{2t}) - \frac{n}{2t} + \frac{1}{t}.$$

The right hand side of the inequality is maximized when $n = tk + t - t^2$ and since $k \geq t$, we get $n \geq 1$. Reporting this value in the above inequation gives

$$rl_k(D_{tk+t-t^2}(1, t)) \geq (tk + t - t^2 - 1)(k + 1 - t - \frac{tk + t - t^2}{2t}) - \frac{tk + t - t^2}{2t} + \frac{1}{t}.$$

After simplification, we obtain

$$rl_k(D_{tk+t-t^2}(1, t)) \geq \frac{t}{2}k^2 - (t^2 - t + 1)k + \frac{t^3}{2} - t^2 + \frac{3}{2}t - 1 + \frac{1}{t},$$

which concludes the proof. \square

Corollary 2. *For any $k \geq 3$,*

$$rl_k(D(2, 3)) \geq \frac{3}{2}k^2 - 7k + 8.$$

3 Upper bounds

3.1 $D(1, 2, \dots, t)$

Recall that a labeling c of vertices of a graph G is a radio k -labeling if, for every pair i, j of vertices of G ,

$$|c_i - c_j| + d_G(i, j) > k, \quad (1)$$

where c_i, c_j denote labels of i and j respectively.

Lemma 5. *Let $t \geq 2$ be an integer, i, j a pair of vertices of the graph $G = D(1, 2, \dots, t)$ and let $|i - j| = qt + r$, where $q, r \in \mathbb{N}$, $0 \leq r < t$. Then $d_G(i, j) = q$ if $r = 0$ and $d_G(i, j) = q + 1$ otherwise.*

Proof. Suppose that $j > i$. There is a path $P = i, i + t, i + 2t, \dots, i + qt$ of length q in G . If $r = 0$ then $j = i + qt$ and hence $d_G(i, j) \leq q$, else there is an edge between vertices $i + qt, i + qt + r = j$ in G and hence $d_G(i, j) \leq q + 1$. Clearly there is no shorter i, j -path in G . \square

Theorem 2. *Let $t \in \mathbb{N}$, k be an even positive integer and let $G = D(1, 2, \dots, t)$. Then*

$$rl_k(G) \leq \frac{t}{2}k^2 + k.$$

Moreover, there is the following periodic pattern:

$$0, k, 2k, 3k, \dots, \left(\frac{t}{2}k + 1\right)k, \frac{k}{2}, \frac{k}{2} + k, \frac{k}{2} + 2k, \dots, \frac{k}{2} + \left(\frac{t}{2}k\right)k. \quad (2)$$

Proof. We show that a labeling c defined by periodical repeating of the given pattern is a radio k -labeling of G , i.e., the inequality (1) holds for every $i, j \in V(G)$. Note that the length of the pattern is $\frac{t}{2}k + 2 + \frac{t}{2}k + 1 = tk + 3$. For every pair i, j of vertices of G with $|i - j| \geq tk + 3$, it holds that $d_G(i, j) > k$. Therefore it suffices to prove that there is no collision in labeling c between vertices in two consecutive copies of the pattern. If $|c_i - c_j| > k$ then the inequality (1) trivially holds. Now we consider the following possibilities.

Case 1: $|c_i - c_j| = 0$.

By the definition of the pattern (2) it follows that $|i - j| = tk + 3$. Then $d_G(i, j) > k$, implying that the inequality (1) holds.

Case 2: $|c_i - c_j| = k$.

Then trivially $|i - j| > 0$ and hence $d_G(i, j) > 0$. This implies that the inequality (1) holds.

Case 3: $0 < |c_i - c_j| < k$.

From pattern (2) we obtain $|c_i - c_j| = \frac{k}{2}$, and $|i - j| = \frac{t}{2}k + 1$ or $|i - j| = \frac{t}{2}k + 2$. Then, by Lemma 5, $d_G(i, j) > \frac{k}{2}$. Thus we have $|c_i - c_j| + d_G(i, j) > \frac{k}{2} + \frac{k}{2} = k$ and the inequality (1) holds.

We have shown that the defined labeling is a radio k -labeling of G . The maximum used label is $\frac{t}{2}k^2 + k$. \square

Theorem 3. *Let $t \geq 2$ be a positive integer, k be an odd positive integer and let $G = D(1, 2, \dots, t)$. Then*

$$rl_k(G) \leq \frac{t}{2}k^2 + \frac{t}{2}k.$$

Moreover, there are the following periodic patterns (where $l = k + 1$):

$$0, l, 2l, 3l, \dots, (\frac{t}{2}k)l, \frac{l}{2}, \frac{l}{2} + l, \frac{l}{2} + 2l, \dots, \frac{l}{2} + (\frac{t}{2}k - 1)l \quad \text{when } t \text{ is even,} \quad (3)$$

$$0, l, 2l, 3l, \dots, (\frac{t}{2}k - \frac{1}{2})l, \frac{l}{2}, \frac{l}{2} + l, \frac{l}{2} + 2l, \dots, \frac{l}{2} + (\frac{t}{2}k - \frac{1}{2})l \quad \text{when } t \text{ is odd.} \quad (4)$$

Proof. We show that a labeling c defined by periodical repeating of the given pattern is a radio k -labeling of G , i.e., the inequality (1) holds for every $i, j \in V(G)$. Note that the length of the pattern is $\frac{t}{2}k + 1 + \frac{t}{2}k = tk + 1$ for even t , and $\frac{t}{2}k + \frac{1}{2} + \frac{t}{2}k + \frac{1}{2} = tk + 1$ for odd t . For every pair i, j of vertices of G with $|i - j| \geq tk + 1$, it holds that $d_G(i, j) > k$. Therefore it suffices to prove that there is no collision in labeling c between vertices in two consecutive copies of the pattern. If $|c_i - c_j| > k$ then the inequality (1) trivially holds. Now we consider the following possibilities.

Case 1: $|c_i - c_j| = 0$.

By the definition of the patterns (3) and (4) it follows that $|i - j| = tk + 1$. Thus $d_G(i, j) > k$, implying that the inequality (1) holds.

Case 2: $|c_i - c_j| = k$.

Then trivially $|i - j| > 0$ and hence $d_G(i, j) > 0$. This implies that the inequality (1) holds.

Case 3: $0 < |c_i - c_j| < k$.

From the patterns (3) and (4) we obtain $|c_i - c_j| = \frac{l}{2}$. Now we have two subcases depending on parity of t .

Subcase 3.1: t is even.

From pattern (3) we get $|i - j| = \frac{t}{2}k = \frac{k-1}{2}t + \frac{t}{2}$ or $|i - j| = \frac{t}{2}k + 1 = \frac{k-1}{2}t + \frac{t}{2} + 1$. Then, by Lemma 5, $d_G(i, j) > \frac{k-1}{2}$. Since $l = k+1$, we have $|c_i - c_j| + d_G(i, j) > \frac{k+1}{2} + \frac{k-1}{2} = k$ and the inequality (1) holds.

Subcase 3.2: t is odd.

From pattern (4) we have $|i - j| = \frac{t}{2}k + \frac{1}{2} = \frac{k-1}{2}t + \frac{t}{2} + \frac{1}{2}$ or $|i - j| = \frac{t}{2}k + \frac{3}{2} = \frac{k-1}{2}t + \frac{t}{2} + \frac{3}{2}$ or (for $t > 1$) $|i - j| = \frac{t}{2}k - \frac{1}{2} = \frac{k-1}{2}t + \frac{t}{2} - \frac{1}{2}$. In each of these possibilities we have $d_G(i, j) > \frac{k-1}{2}$ by Lemma 5. Since $l = k+1$, we have $|c_i - c_j| + d_G(i, j) > \frac{k+1}{2} + \frac{k-1}{2} = k$ and the inequality (1) holds.

We have shown that the defined labeling is a radio k -labeling of G . The maximum used label is $\frac{t}{2}k^2 + \frac{t}{2}k$. \square

We end this subsection by presenting some lower and upper bounds for $\text{rl}_k(D(1, 2, \dots, t))$ for small positive integers k and t (see Table 1). The emphasized numbers are exact values, all the pairs of values are lower and upper bounds.

$t \setminus k$	2	3	4	5	6	7	8	9
2	6	12	20	30	42	56	65 – 72	82 – 90
3	8	17	28	43	55 – 60	74 – 81	97 – 104	122 – 135
4	10	22	36	51 – 56	73 – 78	99 – 112	129 – 136	163 – 180
5	12	27	43	63 – 69	91 – 96	123 – 131	161 – 168	203 – 217
6	14	32	49 – 52	76 – 82	109 – 114	148 – 163	193 – 200	244 – 259
7	16	32 – 37	57 – 60	88 – 95	127 – 132	172 – 189	225 – 232	284 – 301
8	18	37 – 42	65 – 68	101 – 108	145 – 150	197 – 215	257 – 264	325 – 343
9	20	41 – 47	73 – 76	113 – 121	163 – 168	221 – 241	289 – 296	365 – 385

Table 1: Values and bounds of rl_k for few distance graphs $D(1, 2, \dots, t)$

3.2 $D(1, t)$

Now we focus on the distance graphs $G = D(1, t)$. For even k we did not find any improvement of the upper bound for $rl_k(G)$ given by Theorem 2, but for odd k we can improve the upper bound for $rl_k(G)$ from Theorem 3.

Theorem 4. *Let $t \geq 3$ and $k \geq 1$ be odd integers, let G be a distance graph $D(1, t)$. Then $rl_k(G) \leq \frac{t}{2}k^2 - \frac{1}{2}$. Moreover there is the following periodic pattern:*

$$0, k, 2k, 3k, \dots, \left(\frac{t}{2}k - \frac{1}{2}\right)k, \frac{k-1}{2}, \frac{k-1}{2} + k, \frac{k-1}{2} + 2k, \dots, \frac{k-1}{2} + \left(\frac{t}{2}k - \frac{1}{2}\right)k. \quad (5)$$

Proof. We show that a labeling c defined by periodical repeating of the given pattern is a radio k -labeling of G , i.e., the inequality (1) holds for every $i, j \in V(G)$. Note that the length of the pattern is $tk + 1$ and clearly $d_G(i, j) > k$ for every i, j with $|i - j| \geq tk + 1$. Therefore it suffices to prove that there is no collision in labeling c between vertices in two consecutive copies of the pattern. If $|c_i - c_j| > k$ then the inequality (1) trivially holds. Now we consider the following possibilities.

Case 1: $|c_i - c_j| = 0$. From the definition of the pattern (5) it follows that $|i - j| = tk + 1$.

By Lemma 3, $d_G(i, j) > k$ and we are done.

Case 2: $|c_i - c_j| = k$. Clearly $d_G(i, j) > 0$ and then $|c_i - c_j| + d_G(i, j) > k$.

Case 3: $0 < |c_i - c_j| < k$. From (5) we have $c_j = c_i \pm \frac{k \pm 1}{2}$. We consider the following possibilities.

a) $|c_i - c_j| = \frac{k-1}{2}$. From (5) we obtain $|i - j| = \frac{t}{2}k + \frac{1}{2} = \frac{k-1}{2}t + \frac{t}{2} + \frac{1}{2}$. From Lemma 3 for $q = \frac{k-1}{2}$ and $r = \frac{t}{2} + \frac{1}{2}$, we get $d_G(i, j) = \min \left\{ \frac{k-1}{2} + \frac{t}{2} + \frac{1}{2}; \frac{k-1}{2} + 1 + t - \left(\frac{t}{2} + \frac{1}{2}\right) \right\}$. Then $d_G(i, j) > \frac{k+1}{2}$ for $t > 1$. Thus we have $|c_i - c_j| + d_G(i, j) > \frac{k-1}{2} + \frac{k+1}{2} = k$.

b) $|c_i - c_j| = \frac{k+1}{2}$. From (5) we have $|i - j| = \frac{t}{2}k + \frac{1}{2} \pm 1$.

- $|i - j| = \frac{t}{2}k + \frac{1}{2} - 1 = \frac{k-1}{2}t + \frac{t}{2} - \frac{1}{2}$. By Lemma 3 for $q = \frac{k-1}{2}$ and $r = \frac{t}{2} - \frac{1}{2}$, it follows that $d_G(i, j) = \min \left\{ \frac{k-1}{2} + \frac{t}{2} - \frac{1}{2}; \frac{k-1}{2} + 1 + t - \left(\frac{t}{2} - \frac{1}{2}\right) \right\}$. Hence we have $d_G(i, j) > \frac{k-1}{2}$ for $t > 1$.

- $|i - j| = \frac{t}{2}k + \frac{1}{2} + 1 = \frac{k-1}{2}t + \frac{t}{2} + \frac{3}{2}$. By Lemma 3 for $q = \frac{k-1}{2}$ and $r = \frac{t}{2} + \frac{3}{2}$, it follows that $d_G(i, j) = \min \left\{ \frac{k-1}{2} + \frac{t}{2} + \frac{3}{2}; \frac{k-1}{2} + 1 + t - \left(\frac{t}{2} + \frac{3}{2}\right) \right\}$. Thus $d_G(i, j) > \frac{k-1}{2}$ for $t > 1$.

Now we have $|c_i - c_j| + d_G(i, j) > \frac{k+1}{2} + \frac{k-1}{2} = k$ and we are done.

We have shown that the defined labeling is a radio k -labeling of G . The maximum used label is $\frac{k-1}{2} + \left(\frac{t}{2}k - \frac{1}{2}\right)k = \frac{t}{2}k^2 - \frac{1}{2}$. \square

Theorem 5. Let $t \geq 4$ be an even integer and $k \geq 1$ be an odd integer, let G be a distance graph $D(1, t)$. Then $rl_k(G) \leq \frac{t}{2}k^2$. Moreover there is the following periodic pattern:

$$0, k, 2k, 3k, \dots, \left(\frac{t}{2}k\right)k, \frac{k-1}{2}, \frac{k-1}{2} + k, \frac{k-1}{2} + 2k, \dots, \frac{k-1}{2} + \left(\frac{t}{2}k - 1\right)k \quad (6)$$

Proof. We show that the labeling c of G defined by periodical repeating of the given pattern is a radio k -labeling of G , i.e., the inequality (1) holds for every $i, j \in V(G)$. Note that the length of the pattern is $tk + 1$ and clearly $d_G(i, j) > k$ for every $i, j \in \mathbb{Z}$ with $|i - j| \geq tk + 1$. Therefore it suffices to prove that there is no collision in labeling c between vertices in two consecutive copies of the pattern. If $|c_i - c_j| > k$ then the inequality (1) trivially holds. Now we consider the following possibilities.

Case 1: $|c_i - c_j| = 0$. From the definition of the pattern (6) it follows that $|i - j| = tk + 1$. Then $d_G(i, j) > k$ by Lemma 3 and the inequality (1) holds.

Case 2: $|c_i - c_j| = k$. Clearly $d_G(i, j) > 0$ and then $|c_i - c_j| + d_G(i, j) > k$.

Case 3: $0 < |c_i - c_j| < k$. From (6) we have $c_j = c_i \pm \frac{k \pm 1}{2}$, i.e., $|c_i - c_j| = \frac{k \pm 1}{2}$. From (6) we also obtain $|i - j| = \frac{t}{2}k = \frac{k-1}{2}t + \frac{t}{2}$ or $|i - j| = \frac{t}{2}k + 1 = \frac{k-1}{2}t + \frac{t}{2} + 1$. Suppose that $|i - j| = \frac{k-1}{2}t + \frac{t}{2}$. Then, by Lemma 3, $d_G(i, j) = \min\left\{\frac{k-1}{2} + \frac{t}{2}; \frac{k-1}{2} + 1 + t - \frac{t}{2}\right\}$. For $t \geq 4$ we get $d_G(i, j) > \frac{k+1}{2}$. Now suppose that $|i - j| = \frac{k-1}{2}t + \frac{t}{2} + 1$. Then, by Lemma 3, $d_G(i, j) = \min\left\{\frac{k-1}{2} + \frac{t}{2} + 1; \frac{k-1}{2} + 1 + t - \frac{t}{2} - 1\right\}$. For $t \geq 4$ we get $d_G(i, j) > \frac{k+1}{2}$. Thus we have shown that, for every even $t \geq 4$, $|c_i - c_j| + d_G(i, j) > \frac{k-1}{2} + \frac{k+1}{2} = k$.

We have shown that the defined labeling is a radio k -labeling of G . Clearly the maximum used label is $\left(\frac{t}{2}k\right)k = \frac{t}{2}k^2$. \square

Now we consider the particular case when $t = 2$. Lower and upper bounds for the distance graph $D(1, 2)$ can be derived from radio labelings of the square of paths and cycles. Let P_n^2 and C_n^2 be the square of the path and of the cycle of order n , respectively.

Proposition 4. For any integers $n \geq 3$ and $k \geq 1$,

$$rl_k(P_n^2) \leq rl_k(D(1, 2)) \leq rl_k(C_n^2).$$

Proof. The graph P_n^2 is a subgraph of $D(1, 2)$, hence the left part is satisfied. For the right part, one can use any radio k -labeling of C_n^2 as a pattern to label the distance graph $D(1, 2)$. \square

Corollary 3. For any positive integer k ,

$$k^2 + 2 \leq rl_k(D(1, 2)) \leq k^2 + 2k.$$

Proof. Liu and Xie showed that $rl_k(P_{2k+1}^2) = k^2 + 2$ [11] and $rl_k(C_{4k+1}^2) = k^2 + 2k$ [12]. \square

From Theorem 2 and Theorem 3 we obtain the following statement which strengthens the upper bound given in the previous Corollary.

Corollary 4. *For any positive integer k ,*

$$rl_k(D(1, 2)) \leq k^2 + k.$$

At the end of this subsection we present a table of some lower and upper bounds for $rl_k(D(1, t))$ for small positive integers k and t . The emphasized numbers are exact values, all the pairs of values are lower and upper bounds.

$t \setminus k$	2	3	4	5	6	7	8	9
2	6	12	20	30	42	56	65 – 72	82 – 90
3	6	11	24	33	51 – 52	61 – 73	81 – 100	105 – 121
4	6	15	26	43	54 – 64	69 – 94	95 – 116	124 – 152
5	6	13	26	41	49 – 66	73 – 91	103 – 140	137 – 165
6	7	14	28	41 – 48	46 – 72	73 – 102	105 – 147	144 – 196
7	7	12	26	37	42 – 78	69 – 111	104 – 146	145 – 201
8	7	13	26	36 – 48	46 – 75	62 – 116	98 – 159	141 – 212
9	6	11	25 – 28	32 – 41	40 – 74	54 – 99	89 – 156	134 – 207

Table 2: Values of rl_k for some distance graphs $D(1, t)$

3.3 $D(t - 1, t)$

Now we consider the distance graph $G = D(t - 1, t)$. For even k we did not find any improvement of the upper bound for $rl_k(G)$ given in Theorem 2. But for odd k we prove the following statement which decreases the upper bound for $rl_k(G)$ given by Theorem 3.

Theorem 6. *Let $t > 2$ be an integer and $k \geq 3$ be an odd integer, let G be a distance graph $D(t - 1, t)$. Then $rl_k(G) \leq \frac{t}{2}k^2 + k - \frac{t+2}{2}$. Moreover there is the following periodic pattern of colours on the vertices of G .*

$$\begin{aligned} &0, k - 1, 2(k - 1), 3(k - 1), \dots, \left(\frac{t}{2}k + \frac{t+2}{2}\right)(k - 1), \\ &\frac{k-1}{2}, \frac{k-1}{2} + (k - 1), \frac{k-1}{2} + 2(k - 1), \dots, \frac{k-1}{2} + \left(\frac{t}{2}k + \frac{t}{2}\right)(k - 1) \end{aligned} \quad (7)$$

Proof. We show that a labeling c defined by periodical repeating of the given pattern is a radio k -labeling of G , i.e., the inequality (1) holds for every $i, j \in V(G)$. Note that the length of the pattern is $\frac{t}{2}k + \frac{t}{2} + 2 + \frac{t}{2}k + \frac{t}{2} + 1 = tk + t + 3$ and clearly $d_G(i, j) > k$ for every i, j with $|i - j| \geq tk + t + 3$. Therefore it suffices to prove that there is no collision in labeling c between vertices in two consecutive copies of the pattern. If $|c_i - c_j| > k$ then the inequality (1) trivially holds. The following possibilities can occur.

Case 1: $|c_i - c_j| = 0$. From the definition of the pattern (7), it follows that $|i - j| > tk$, implying that the inequality (1) holds.

Case 2: $|c_i - c_j| = k - 1$. It follows that $|i - j| = 1$. For $t > 2$ we have $d_G(i, j) > 1$, implying that the inequality (1) holds.

Case 3: $|c_i - c_j| = \frac{k-1}{2}$. By the definition of the pattern (7), $|i - j| = \frac{t}{2}k + \frac{t}{2} + 1$ or $|i - j| = \frac{t}{2}k + \frac{t}{2} + 2$. Since $|i - j| > \frac{k+1}{2}t$ and k is odd, we get $d_G(i, j) > \frac{k+1}{2}$, implying that $|c_i - c_j| + d_G(i, j) > \frac{k-1}{2} + \frac{k+1}{2} = k$.

The maximum used label in c is $(\frac{t}{2}k + \frac{t+2}{2})(k-1) = \frac{t}{2}k^2 + k - \frac{t+2}{2}$.

□

We also present the following table with some lower and upper bounds for $\text{rl}_k(D(t-1, t))$ for small positive integers k and t . The emphasized numbers are exact values, all the pairs of values are lower and upper bounds.

$t \setminus k$	2	3	4	5	6	7	8	9
2	6	12	20	30	42	56	65 – 72	82 – 90
3	6	14	28	40	51 – 60	60 – 78	66 – 104	71 – 128
4	7	14	27	40 – 48	50 – 70	57 – 99	65 – 131	71 – 166
5	6	13	26	37 – 50	47 – 78	55 – 116	61 – 150	70 – 191
6	6	14	24 – 30	32 – 54	40 – 69	50 – 108	60 – 153	65 – 208
7	7	13	21 – 28	28 – 48	36 – 74	45 – 109	55 – 148	60 – 223
8	7	13	19 – 26	24 – 48	32 – 73	40 – 120	49 – 148	56 – 211
9	6	12	17 – 27	23 – 53	30 – 70	36 – 109	44 – 164	51 – 233

Table 3: Values of rl_k for some distance graphs $D(t-1, t)$

4 Acknowledgement

Lower and upper bounds on radio k -labeling number of the distance graphs $D(1, 2, \dots, t)$, $D(1, t)$ and $D(t-1, t)$ can be obtained from theorems and propositions given in Sections 2 and 3. For small values $t, k \in \{2, \dots, 9\}$, we improve these bounds using a computer. For finding lower bounds we used brute force search program. The program takes vertices $X = \{1, 2, \dots, i\}$ of the distance graph G and it tries to construct a radio k -labeling c of X using labels $0, \dots, l$. First it assigns label 0 to vertex 1 (there must be a vertex with label 0, otherwise we can decrease all labels to get smaller bound) and tries to extend c to X . If the extension is not possible we conclude that $\text{rl}_k(G) > l$.

For finding upper bounds, we found and verified (again using computer) patterns, which can be periodically repeated for a whole distance graph G .

The lower and upper bounds shown in Tables 1, 2 and 3 are presented at the web pages http://home.zcu.cz/~holubpre/radio_labeling/ and were computed on the Metacentrum computing facilities.

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